BIVARIATE DISTRIBUTIONS

Discrete Distributions
Bivariate Distribution: Discrete

• Let $X$ and $Y$ be two discrete random variables.
• The joint probability mass function $f(x, y)$ is defined for each pair of numbers $(x, y)$ by $f(x, y) = P(X = x \text{ and } Y = y)$.

1. $0 \leq f(x, y) \leq 1$
2. $\sum \sum_{(x,y) \in S} f(x, y) = 1$.
3. $P[(X, Y) \in A] = \sum \sum_{(x,y) \in A} f(x, y)$, where $A \subseteq S$. 
Example:

Consider the following joint probability distribution $p(x, y)$ of two random variables $X$ and $Y$:

<table>
<thead>
<tr>
<th>X \ Y</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.15</td>
<td>0.1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0.25</td>
<td>0.3</td>
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(a) Find $P(X+Y=2)$.
(b) Find $P(X>Y)$. 

Marginal Probability

• The **marginal probability mass functions** of X and of Y are given by

\[ f_X(x) = \sum_y f(x, y), \quad x \in S_x \]

\[ f_Y(y) = \sum_x f(x, y), \quad y \in S_y \]
Example:

- Consider the following joint probability distribution $p(x, y)$ of two random variables $X$ and $Y$:

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c) Find the marginal probability distribution of $X$ and $Y$. 
Independence

• Random variable $X$ and $Y$ are independent if and only if,
• For every $x \in S_x, y \in S_y$,
  
  $$P(X = x, Y = y) = P(X = x)P(Y = y).$$

• Or, equivalently,
  
  $$f(x, y) = f_X(x)f_Y(y).$$

• Or, equivalently,
  
  $$F(x, y) = F_X(x)F_Y(y).$$

• Otherwise, $X$ and $Y$ are said to be dependent.
Example:

• Consider the following joint probability distribution $p(x, y)$ of two random variables $X$ and $Y$:

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• Are $X$ and $Y$ independent?
  • (A) Yes    (B) No
Expectation

• Let $g(X, Y)$ be the function of two random variables $X$ and $Y$.
• Then
  
  $E[g(X, Y)] = \sum\sum_{(x,y) \in S} g(x, y) f(x, y)$.

• Special cases include:
  • $g(X, Y) = X$ or $Y$
  • $g(X, Y) = (X - \mu_X)^2$ or $(Y - \mu_Y)^2$
Example:

Consider the following joint probability distribution $p(x, y)$ of two random variables $X$ and $Y$:

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\[ d) \text{ Find } E[X], E[Y], E[X + Y], E[X \cdot Y]. \]

- (A) 1.75  (B) 0.8  (C) 1.4  (D) 1.5
BIVARIATE DISTRIBUTIONS

Continuous distributions
Bivariate Distribution: Continuous

- Let $X$ and $Y$ be two continuous random variables.
- Then $f(x, y)$ is the **joint probability density function** $f(x, y)$ for $X$ and $Y$ if

1. $f(x, y) \geq 0$
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = 1$
3. For any two-dimensional set $A$, $P((X, Y) \in A) = \iint_{A} f(x, y) \, dx \, dy$. 
Example

• Let random variable $X$ and $Y$ be the random variable on $D = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1, x + y \leq 1\}$.
• Let the joint probability of $(X, Y)$ be

$$f(x, y) = \begin{cases} 60x^2y & 0 \leq x \leq 1, 0 \leq y \leq 1, x + y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

a) Verify that $f(x, y)$ is a legitimate probability density function.
b) Find $P(X + Y < 0.5)$.
c) Find $P(X > Y)$.
d) Find $P(2X \leq Y)$.  
(A) $\frac{10}{27}$  
(B) $\frac{15}{81}$  
(C) $\frac{18}{243}$  
(D) $\frac{1}{9}$
Marginal Probability

- The **marginal probability density functions** of $X$ and of $Y$ are given by

  - $f(x) = \int_{-\infty}^{\infty} f(x, y) \, dy$, $x \in S_x$
  - $f(y) = \int_{-\infty}^{\infty} f(x, y) \, dx$, $y \in S_y$
Example

• Let random variable X and Y be the random variable on
  \[ D = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1, x + y \leq 1\}. \]
• Let the joint probability of (X, Y) be
  \[
  f(x, y) = \begin{cases} 
    60x^2y & 0 \leq x \leq 1, 0 \leq y \leq 1, x + y \leq 1 \\
    0 & \text{otherwise.}
  \end{cases}
  \]

(e) Find the marginal probability density function for X.
(f) Find the marginal probability density function for Y.
Expectation

- Let $g(X, Y)$ be the function of two random variables $X$ and $Y$.
- Then
  \[ E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)f(x, y)\,dx\,dy. \]

- Again, special cases include:
  - $g(X, Y) = X$ or $Y$
  - $g(X, Y) = (X - \mu_X)^2$ or $(Y - \mu_Y)^2$
Example

- Let random variable X and Y be the random variable on
- \( D = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1, x + y \leq 1\} \).
- Let the joint probability of (X,Y) be
- \( f(x, y) = \begin{cases} 
60x^2y & 0 \leq x \leq 1, 0 \leq y \leq 1, x + y \leq 1 \\
0 & \text{otherwise.}
\end{cases} \)

\( g) \) Find \( E[X], E[Y], E[X + Y], E[X \ast Y] \).

\( h) \) (A) \( \frac{5}{6} \)  (B) \( \frac{1}{2} \)  (C) \( \frac{1}{3} \)  (D) 1
Independence

• Random variable X and Y are independent if and only if,
• For every $x \in S_x, y \in S_y$,
  $f(x, y) = f_X(x)f_Y(y)$.
• Or, equivalently,
  $F(x, y) = F_X(x)F_Y(y)$.

• Otherwise, X and Y are said to be dependent.
Example

- Let random variable $X$ and $Y$ be the random variable on $D = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1, x + y \leq 1\}$.
- Let the joint probability of $(X,Y)$ be

$$f(x, y) = \begin{cases} 
60x^2y & 0 \leq x \leq 1, 0 \leq y \leq 1, x + y \leq 1 \\
0 & \text{otherwise.}
\end{cases}$$

h) Are random variables $X$ and $Y$ independent?
Independence and Expectation

- If random variables $X$ and $Y$ are independent, then
- $E[g(X) \cdot h(Y)] = E[g(X)] \cdot E[h(Y)]$. 
Independence and Expectation

- Suppose the probability density functions of $T_1$ and $T_2$ are
  \[ f_{T_1}(x) = \alpha e^{-\alpha x}, x > 0, \]
  \[ f_{T_2}(y) = \beta e^{-\beta y}, y > 0. \]
- Suppose $T_1$ and $T_2$ are independent. Find $E[T_1 T_2]$. 
Read Assignments

• Please Read 4.2, 4.3